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The Whittaker Constant and Successive Derivatives of Entire Functions

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1. INTRODUCTION

For an entire function f with maximum modulus

$$M(r) = M(r;f) = \max_{|z|=r} |f(z)|,$$

the *exponential type* $\tau(f)$ is given by

$$\tau(f) = \limsup_{n\to\infty} |f^{(n)}(0)|^{1/n} = \limsup_{r\to\infty} \frac{\log M(r)}{r}.$$

The Whittaker constant W is defined to be the greatest positive number c with the following property: If $\tau(f) < c$ and each of $f, f', f'',...$ has a zero in the closed disc $|z| \leqslant 1$, then $f \equiv 0$. The numerical value of W is known to lie between .7259 and .7378 [6], [7]. The conjecture $W = 2/e$ has remained unsettled since 1943 [2].

An exact determination of W was obtained by M. A. Evgrafov [3] in 1954. The determination involves the Gončarov polynomials $G_n(z; z_0,..., z_{n-1})$ defined recursively by

$$G_0(z) = 1,$$

$$G_n(z; z_0,..., z_{n-1}) = \frac{z^n}{n!} - \sum_{k=0}^{n-1} \frac{z_k^{n-k}}{(n-k)!} G_k(z; z_0,..., z_{k-1}).$$

Let

$$H_n = \max | G_n(0; z_0,..., z_{n-1})|,$$

where the maximum is taken over all sequences $\{z_k\}_0^{n-1}$ whose terms lie on $|z| = 1$. Evgrafov proved that

$$W = \{\limsup_{n\to\infty} H_n^{1/n}\}^{-1}.$$

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In the present paper we improve Evgrafov's result and obtain a second characterization of W. For $n = 1, 2, 3,...$ and $0 \leqslant u < \infty$, let

$$T_n(u) = \max \sum_{k=n}^{\infty} u^k \mid G_k(0; z_0, ..., z_{n-1}, 0, ..., 0)|,$$

where the maximum is taken over all sequences $\{z_k\}_0^{n-1}$ whose terms lie on $|z| = 1$. Let u_n, $u_n > 0$, be such that $T_n(u_n) = 1$, $n = 1, 2, 3,...$. We prove the following:

$$W = \{\lim_{n\to\infty} H_n^{1/n}\}^{-1} = \{\sup_{1\leqslant n<\infty} H_n^{1/n}\}^{-1} \tag{1.1}$$

and

$$W = \lim_{n\to\infty} u_n = \sup_{1\leqslant n<\infty} u_n . \tag{1.2}$$

These are consequences of the estimates

$$(.4)^{1/n} H_n^{-1/n} < W \leqslant H_n^{-1/n} \tag{1.3}$$

and

$$u_n \leqslant W < (1.6)^{1/n} u_n \tag{1.4}$$

which hold for all positive integers n. On the basis of either (1.3) or (1.4), the constant W can (in theory if not in practice) be calculated as accurately as desired.

There are two matters related to the Whittaker constant that are of considerably more importance than its numerical value. The first, which is due to Evgrafov, is the existence of extremal functions.

THEOREM A (Evgrafov). *There exists an entire function F of exponential type W such that each of $F, F', F'',...$ has a zero in the* disc $|z| \leqslant 1$.

The second is a coefficient inequality which yields considerable information about zeros of successive derivatives.

THEOREM B. *Suppose n is a positive integer, and f is analytic in $|z| \leqslant 1$. If each of $f, f',..., f^{(n-1)}$ has a zero in $|z| \leqslant 1$, then*

$$|f(0)| \leqslant \frac{1.1}{W^n} \sum_{m=0}^{\infty} \frac{|f^{(n+m)}(0)|}{(m+1)!} . \tag{1.5}$$

Furthermore, there is an entire function f with the property that each of $f, f', f'',...$ has a zero in $|z| \leqslant 1$ and such that

$$|f(0)| > \frac{.67}{W^n} \sum_{m=0}^{\infty} \frac{|f^{(n+m)}(0)|}{(m+1)!}, \qquad n = 1, 2, 3,... .$$

Suppose that f is entire, $\tau(f) < W$, and each of $f, f', f'',...$ has a zero in $|z| \leqslant 1$. One can argue directly from Theorem B that $f \equiv 0$. The condition

$\tau(f) < W$ assures that the right member of (1.5) approaches 0 as $n \to \infty$; therefore $f(0) = 0$. By applying Theorem B to each derivative of f we obtain $f^{(j)}(0) = 0, j = 1, 2, 3, \ldots$, so that $f \equiv 0$. Taken together, Theorems A and B give a fairly complete description of an interesting property of entire functions of exponential type.

A related and equally interesting problem concerns univalence of successive derivatives of entire functions. R. P. Boas [1] proved that if f is a transcendental entire function such that $\tau(f) < \log 2$, then infinitely many derivatives of f are univalent in $|z| \leqslant 1$. Levinson [4] obtained a simpler proof of Boas' result, but his method of proof affords no improvement on the constant log 2. In view of Theorem A, the constant log 2 can not replaced by a number greater than W. Using a univalent analogue of Theorem B, we prove the following result.

THEOREM C. *Let f be a transcendental entire function whose exponential type is less than W. If D is a closed* disc *of radius* 1, *then infinitely many derivatives of f are univalent in D.*

All the properties of Gončarov polynomials which we use are developed in Section 2. This seemed desirable since Evgrafov's work [3] is available only in Russian. The results contained in Lemmas 1 and 4, Theorems 1, 2, and 3, and Corollary 1 are known and can be found in [3].

2. GONČAROV POLYNOMIALS

Suppose f is an entire function and $\{z_k\}_0^\infty$ a sequence of complex numbers. If we write the defining relation for Gončarov polynomials in the form

$$\frac{z^n}{n!} = \sum_{k=0}^{n} \frac{z_k^{n-k}}{(n-k)!} G_k(z; z_0, \ldots, z_{k-1}),$$

we have

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} f^{(n)}(0) \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} f^{(n)}(0) \sum_{k=0}^{n} \frac{z_k^{n-k}}{(n-k)!} G_k(z; z_0, \ldots, z_{k-1}) \\ &= \sum_{k=0}^{\infty} G_k(z; z_0, \ldots, z_{k-1}) \sum_{n=k}^{\infty} \frac{f^{(n)}(0)\, z_k^{n-k}}{(n-k)!} \\ &= \sum_{k=0}^{\infty} f^{(k)}(z_k)\, G_k(z; z_0, \ldots, z_{k-1}) \qquad (2.1) \end{aligned}$$

whenever the interchange in order of summation can be justified. In particular, (2.1) holds if f is a polynomial.

LEMMA 1. *The polynomials G_n have the following properties*:

$$G_n(\lambda z; \lambda z_0, \ldots, \lambda z_{n-1}) = \lambda^n G_n(z; z_0, \ldots, z_{n-1}); \tag{2.2}$$

$$G_n(z_0; z_0, \ldots, z_{n-1}) = 0 \qquad (n > 0); \tag{2.3}$$

$$G_n'(z; z_0, \ldots, z_{n-1}) = G_{n-1}(z; z_1, \ldots, z_{n-1}) \qquad (n > 0). \tag{2.4}$$

Proof. Mathematical induction. In (2.4) the indicated differentiation is with respect to z.

As an immediate consequence of (2.4), we have

$$G_n^{(k)}(z; z_0, \ldots, z_{n-1}) = G_{n-k}(z; z_k, \ldots, z_{n-1}), \qquad 0 \leqslant k \leqslant n,$$

and from (2.3),

$$G_n^{(k)}(z_k; z_0, \ldots, z_{n-1}) = 0, \qquad k = 0, 1, \ldots, (n-1).$$

The last equation, together with $G_n^{(n)}(z; z_0, \ldots, z_{n-1}) \equiv 1$, completely determine the Gončarov polynomials, and allow one to express G_n (as Gončarov did originally) as an iterated integral,

$$G_n(z; z_0, \ldots, z_{n-1}) = \int_{z_0}^{z} \int_{z_1}^{x_0} \cdots \int_{z_{n-1}}^{x_{n-2}} dx_{n-1}\, dx_{n-2} \cdots dx_0 .$$

Algebraic properties of the Gončarov polynomials are, for the most part, special cases of an algebraic identity which itself is a special case of (2.1). In (2.1), replace $\{z_k\}_0^\infty$ by a sequence $\{w_k\}_0^\infty$ and replace f by the polynomial $G_n(z; z_0, \ldots, z_{n-1})$. This yields

$$\begin{aligned} G_n(z; z_0, \ldots, z_{n-1}) &= \sum_{k=0}^{\infty} G_n^{(k)}(w_k; z_0, \ldots, z_{n-1})\, G_k(z; w_0, \ldots, w_{k-1}) \\ &= \sum_{k=0}^{n} G_{n-k}(w_k; z_k, \ldots, z_{n-1})\, G_k(z; w_0, \ldots, w_{k-1}). \end{aligned} \tag{2.5}$$

The numbers w_k in (2.5) are arbitrary; if we take them all to be 0, we obtain

$$G_n(z; z_0, \ldots, z_{n-1}) = \sum_{k=0}^{n} G_{n-k}(0; z_k, \ldots, z_{n-1}) \frac{z^k}{k!} . \tag{2.6}$$

The other special case of (2.5) which we shall need is the following: let m be an integer such that $0 < m < n$ and let

$$w_k = \begin{cases} z_k, & 0 \leqslant k < m \\ 0, & m \leqslant k < n. \end{cases}$$

Then

$$G_n(z; z_0, ..., z_{n-1}) = \sum_{k=0}^{m-1} G_{n-k}(z_k; z_k, ..., z_{n-1}) G_k(z; z_0, ..., z_{k-1})$$

$$+ \sum_{k=m}^{n} G_{n-k}(0; z_k, ..., z_{n-1}) G_k(z; z_0, ..., z_{m-1}, 0, ..., 0)$$

$$= \sum_{k=m}^{n} G_{n-k}(0; z_k, ..., z_{n-1}) G_k(z; z_0, ..., z_{m-1}, 0, ..., 0).$$

Replacing k by $n - k$, we obtain

$$G_n(z; z_0, ..., z_{n-1}) = \sum_{k=0}^{n-m} G_k(0; z_{n-k}, ..., z_{n-1}) G_{n-k}(z; z_0, ..., z_{m-1}, 0, ..., 0). \tag{2.7}$$

With obvious notational conventions, (2.7) also holds for $m = 0$ and $m = n$. The importance of (2.7) is that it is a separation of variables formula; the first factors on the right involve only $z_m, ..., z_{n-1}$, and the second factors on the right involve only $z_0, ..., z_{m-1}$. This is crucial for the following lemma.

LEMMA 2. *If* $0 \leqslant m \leqslant n$, *then* $H_n \geqslant H_{n-m} H_m$.

Proof. The result is trivial if $m = 0$ or $m = n$. Suppose $0 < m < n$ and choose the sequence $\{z_k\}_0^{n-1}$ with $|z_k| = 1$, $0 \leqslant k < n$, so that $H_m = |G_m(0; z_0, ..., z_{m-1})|$ and $H_{n-m} = |G_{n-m}(0; z_m, ..., z_{n-1})|$. Clearly,

$$H_n \geqslant \max_{|\lambda|=1} |G_n(0; \lambda z_0, ..., \lambda z_{m-1}, z_m, ..., z_{n-1})|.$$

From (2.7), with $z = 0$, we have

$$G_n(0; \lambda z_0, ..., \lambda z_{m-1}, z_m, ..., z_{n-1})$$

$$= \sum_{k=0}^{n-m} G_k(0; z_{n-k}, ..., z_{n-1}) \lambda^{n-k} G_{n-k}(0; z_0, ..., z_{m-1}, 0, ..., 0)$$

$$= \lambda^m Q(\lambda),$$

where $Q(\lambda)$ is a polynomial in λ. Now

$$H_n \geqslant \max_{|\lambda|=1} |\lambda^m Q(\lambda)| = \max_{|\lambda|=1} |Q(\lambda)| \geqslant |Q(0)|,$$

and

$$\begin{aligned} |Q(0)| &= |G_{n-m}(0; z_m, \dots, z_{n-1})\, G_m(0; z_0, \dots, z_{m-1})| \\ &= H_{n-m} H_m, \end{aligned}$$

which completes the proof.

LEMMA 3.

$$\lim_{n\to\infty} H_n^{1/n} = \sup_{1\leqslant j<\infty} H_j^{1/j}.$$

Proof. Let $j \geqslant 1$ be fixed and write $n = qj + d$, $0 \leqslant d < j$. From Lemma 2 we have

$$H_n \geqslant H_{qj} H_d \geqslant H_j^q H_1^d = H_j^q.$$

Therefore

$$H_n^{1/n} \geqslant H_j^{(n-d)/jn} = H_j^{1/j} H_j^{-d/jn}.$$

Since the last factor approaches 1 as $n \to \infty$, we have

$$\liminf_{n\to\infty} H_n^{1/n} \geqslant H_j^{1/j}.$$

Therefore

$$\liminf_{n\to\infty} H_n^{1/n} \geqslant \sup_{1\leqslant j<\infty} H_j^{1/j},$$

which completes the proof.

LEMMA 4. *For each non negative integer n,*

$$H_n \leqslant (1/\log 2)^n.$$

Proof. From the defining relation of the polynomials G_n one obtains

$$H_n \leqslant \sum_{k=0}^{n-1} \frac{H_k}{(n-k)!} = \sum_{k=1}^{n} \frac{H_{n-k}}{k!}.$$

An easy induction argument establishes the desired result.

Let

$$H = \lim_{n\to\infty} H_n^{1/n} = \sup_{1\leqslant j<\infty} H_j^{1/j}$$

and

$$A(z) = \sum_{k=0}^{\infty} \frac{z^k}{H_k k!}.$$

We note for future use that

$$1 = H_1 \leqslant H \leqslant \frac{1}{\log 2}$$

and that $A(z)$ is an entire function of exponential type $1/H$. The importance of the function $A(z)$ is that, apart from a constant factor, it majorizes a large class of Gončarov polynomials. (A function $f(z) = \sum_0^\infty a_n z^n$ is said to be majorized by $g(z) = \sum_0^\infty b_n z^n$ if $| a_n | \leqslant b_n$, $n = 0, 1, 2,\ldots$.)

LEMMA 5. *If* $| z_k | \leqslant 1$, $0 \leqslant k < n$, *then* $G_n(z; z_0 ,\ldots, z_{n-1})$ *is majorized by* $H_n A(z)$.

Proof. We have from (2.6) that

$$G_n(z; z_0 ,\ldots, z_{n-1}) = \sum_{k=0}^{n} G_{n-k}(0; z_k ,\ldots, z_{n-1}) \frac{z^k}{k!}.$$

Since

$$| G_{n-k}(0; z_k ,\ldots, z_{n-1})| \leqslant H_{n-k} \leqslant H_n/H_k ,$$

the result follows. In particular, we note that

$$| G_n(z; z_0 ,\ldots, z_{n-1})| \leqslant H_n A(| z |) \tag{2.8}$$

holds for all z.

The only other inequality for Gončarov polynomials which we shall need is contained in the following lemma.

LEMMA 6. *If* $1 \leqslant n \leqslant m$ *and* $\{z_k\}_0^{n-1}$ *is a sequence of points in* $| z | \leqslant 1$, *then*

$$| G_m(0; z_0 ,\ldots, z_{n-1} , 0,\ldots, 0)| < \frac{H^n\{\exp(1/H) - 1\}}{(m - n + 1)!}.$$

Proof. Since

$$G_m(0; z_0 ,\ldots, z_{n-1} , 0,\ldots, 0) = - \sum_{k=0}^{n-1} \frac{z_k^{m-k}}{(m - k)!} G_k(0; z_0 ,\ldots, z_{k-1}),$$

its absolute value does not exceed

$$\sum_{k=0}^{n-1} \frac{H_k}{(m-k)!} \leqslant \sum_{k=0}^{n-1} \frac{H^k}{(m-k)!}.$$

Replacing k by $n-k$ in the last sum, we obtain

$$\sum_{k=1}^{n} \frac{H^{n-k}}{(m-n+k)!} < \frac{H^n}{(m-n+1)!} \sum_{k=1}^{\infty} H^{-k} \frac{(m-n+1)!}{(m-n+k)!}$$

$$\leqslant \frac{H^n}{(m-n+1)!} \sum_{k=1}^{\infty} \frac{H^{-k}}{k!},$$

which completes the proof.

3. Entire Functions of Exponential Type

We are now in a position to establish the expansion (2.1) for a large class of functions. Although the following theorem is relatively well-known, our proof is new.

THEOREM 1. *If f is an entire function of exponential type less than $1/H$ and $\{z_k\}_0^\infty$ is a sequence of points in the* disc $|z| \leqslant 1$, *then*

$$f(z) = \sum_{k=0}^{\infty} f^{(k)}(z_k)\, G_k(z; z_0, \ldots, z_{k-1})$$

for all z.

Proof. We need only show that the interchange in order of summation in (2.1) is justified in this case. This will be so provided that the series

$$\sum_{n=0}^{\infty} |f^{(n)}(0)| \sum_{k=0}^{n} \left| \frac{z_k^{n-k}}{(n-k)!} G_k(z; z_0, \ldots, z_{k-1}) \right| \tag{3.1}$$

is convergent. From (2.8) we have

$$|G_k(z; z_0, \ldots, z_{k-1})| \leqslant H_k A(|z|)$$

$$\leqslant \frac{H_n}{H_{n-k}} A(|z|)$$

for $n \geqslant k$. Therefore

$$\sum_{k=0}^{n} \left| \frac{z_k^{n-k}}{(n-k)!} G_k(z; z_0, ..., z_{k-1}) \right| \leqslant H_n A(|z|) \sum_{k=0}^{n} \frac{1}{(n-k)!\, H_{n-k}}$$

$$= H_n A(|z|) \sum_{k=0}^{n} \frac{1}{H_k k!}$$

$$< H_n A(|z|)\, A(1).$$

Therefore (3.1) converges provided that

$$\sum_{n=0}^{\infty} |f^{(n)}(0)|\, H_n \tag{3.2}$$

converges. To establish convergence of (3.2), we use the root test. Since f is of exponential type less than $1/H$, we have

$$\limsup_{n\to\infty} \{|f^{(n)}(0)|\, H_n\}^{1/n} = H \limsup_{n\to\infty} |f^{(n)}(0)|^{1/n}$$
$$< 1.$$

Therefore (3.2) converges and the proof is complete.

As an immediate consequence of Theorem 1 we note that if f is of exponential type less than $1/H$ and $f^{(k)}(z_k) = 0$, $k = 0, 1, 2,...$ for a sequence of points $\{z_k\}_0^\infty$ in $|z| \leqslant 1$, then $f \equiv 0$. It follows from this that the Whittaker constant is at least as great as $1/H$. To complete the proof that $W = 1/H$ we follow the method of Evgrafov and construct an entire function of exponential type $1/H$ such that it and each of its derivatives have a zero on the circle $|z| = 1$.

THEOREM 2. *There is an entire function F of exponential type $1/H$ such that each of $F, F', F'',...$ has a zero on the circle $|z| = 1$.*

Proof. For each positive integer n choose complex numbers $z_k = z_k(n)$, $0 \leqslant k < n$, on the unit circle such that

$$H_n = |G_n(0; z_0, ..., z_{n-1})|.$$

Let

$$P_n(z) = \frac{G_n(z; z_0, ..., z_{n-1})}{G_n(0; z_0, ..., z_{n-1})}.$$

The polynomials P_n satisfy $P_n(0) = 1$ and are majorized by A. Therefore they are uniformly bounded on bounded sets, and one can select a subsequence

$\{P_{n_m}\}$ that converges uniformly on bounded sets to an entire function F, with $F(0) = 1$, which is majorized by A. Therefore F is of exponential type $1/H$ or less.

Since $F \not\equiv 0$, it follows from Hurwitz' theorem that there is a point z_0', $|z_0'| = 1$, such that $F(z_0') = 0$. From the facts that $F(0) = 1$ and $F(z_0') = 0$, it follows that $F' \not\equiv 0$. We can therefore apply Hurwitz' theorem to the sequence $\{P'_{n_m}\}$ and obtain a point z_1' on $|z| = 1$ such that $F'(z_1') = 0$. Since $F' \not\equiv 0$ and $F'(z_1') = 0$, we conclude that F' is nonconstant, and therefore that $F'' \not\equiv 0$. Applying the same argument, we obtain a point z_2' on $|z| = 1$ such that $F''(z_2') = 0$. Continuing in the same manner, for each positive integer k we obtain a point z_k' on $|z| = 1$ such that $F^{(k)}(z_k') = 0$.

All that remains to prove is that the exponential type of F is not less than $1/H$. If it were, Theorem 1 would apply, and the expansion

$$F(z) = \sum_{k=0}^{\infty} F^{(k)}(z_k')\, G_k(z; z_0',..., z'_{k-1})$$

would yield $F \equiv 0$, which is false.

COROLLARY 1. $W = 1/H$.

Entire functions f such that $\tau(f) = W$ and each of $f, f', f'',...$ has a zero in $|z| \leqslant 1$ will be called *Whittaker functions.* The function F of Theorem 2 is a Whittaker function and the derivative of a Whittaker function is a Whittaker function.

THEOREM 3. *There exists a Whittaker function $\mathscr{W}$, with $\mathscr{W}(0) = 1$, which is majorized by e^{Wz}.*

Proof. Let F denote the function of Theorem 2, let $\{t_n\}_1^\infty$ be an increasing sequence of positive numbers with limit 1, and set

$$F_n(z) = F(t_n z).$$

Then $F_n^{(k)}(0) = t_n{}^k F^{(k)}(0)$, and, since F_n is of type less than W, we have

$$\lim_{k\to\infty} \frac{|F_n^{(k)}(0)|}{W^k} = 0.$$

Therefore, there is a positive integer $m = m(n)$ such that

$$\frac{|F_n^{(m)}(0)|}{W^m} > \max_{0<j<\infty} \frac{|F_n^{(m+j)}(0)|}{W^{m+j}}.$$

Let

$$\mathscr{W}_n(z) = F_n^{(m)}(z)/F_n^{(m)}(0) .$$

Then $\mathscr{W}_n(0) = 1$ and

$$| \mathscr{W}_n^{(j)}(0)| = \left| \frac{F_n^{(m+j)}(0)}{F_n^{(m)}(0)} \right| \frac{W^{-m-j}}{W^{-m}} W^j$$
$$\leqslant W^j,$$

so that $\mathscr{W}_n$ is majorized by e^{Wz}. Also, $\mathscr{W}_n$ and each of its derivatives have a zero on the circle $| z | = 1/t_n$. Selecting a uniformly convergent subsequence of $\{\mathscr{W}_n\}$, we obtain a limit function $\mathscr{W}$ with the desired properties.

THEOREM 4. *Suppose n is a positive integer and u is a positive number. If the entire function f is such that each of* $f, f', \ldots, f^{(n-1)}$ *has a zero in* $| z | \leqslant 1$ *and such that*

$$| f^{(k)}(0)| \leqslant u^k \qquad \text{for all} \quad k \geqslant n,$$

then $| f(0)| \leqslant T_n(u)$. *Furthermore, there exists such an f for which* $| f(0)| = T_n(u)$.

Proof. Let $\{z_k\}_0^\infty$ be such that $| z_k | \leqslant 1$ and $f^{(k)}(z_k) = 0$, $0 \leqslant k < n$, and $z_k = 0$, $k \geqslant n$. There is in this case no difficulty in justifying the expansion (2.1), and we have

$$f(z) = \sum_{k=0}^{\infty} f^{(k)}(z_k)\, G_k(z; z_0, \ldots, z_{k-1})$$
$$= \sum_{k=n}^{\infty} f^{(k)}(0)\, G_k(z; z_0, \ldots, z_{n-1}, 0, \ldots, 0). \tag{3.3}$$

Taking $z = 0$, we have

$$| f(0)| = \left| \sum_{k=n}^{\infty} f^{(k)}(0)\, G_k(0, z_0, \ldots, z_{n-1}, 0, \ldots, 0) \right|$$
$$\leqslant \max \left| \sum_{k=n}^{\infty} f^{(k)}(0)\, G_k(0; w_0, \ldots, w_{k-1}, 0, \ldots, 0) \right|,$$

where the maximum is taken over all sequences $\{w_k\}_0^{n-1}$ whose terms lie in $| z | \leqslant 1$. By the maximum modulus theorem we can take, instead, the maximum over all sequences $\{w_k\}_0^{n-1}$ whose terms lie on $|z| = 1$. We then apply the triangle inequality and obtain

$$|f(0)| \leqslant \max \sum_{k=n}^{\infty} |f^{(k)}(0)| \, |G_k(0, w_0, ..., w_{n-1}, 0, ..., 0)|$$

$$\leqslant \max \sum_{k=n}^{\infty} u^k \, |G_k(0; w_0, ..., w_{n-1}, 0, ..., 0)|$$

$$= T_n(u).$$

In passing we note that, for $0 \leqslant j < n$, the function $u^{-j}f^{(j)}$ satisfies the hypotheses of the theorem if n is replaced by $n - j$. We therefore have

$$|f^{(j)}(0)| \leqslant u^j T_{n-j}(u), \qquad j = 0, 1, ..., (n-1),$$

for functions f which satisfy the hypotheses of Theorem 4.

It remains to show that the bound on $f(0)$ is attained. For this purpose, let $\{z_k\}_0^{n-1}$ be a sequence of points on $|z| = 1$ such that

$$T_n(u) = \sum_{k=n}^{\infty} u^k \, |G_k(0; z_0, ..., z_{n-1}, 0, ..., 0)|.$$

Let $\{A_k\}_n^{\infty}$ be a complex sequence determined as follows. For $k \geqslant n$, let $|A_k| = u^k$ and choose the argument of A_k so that

$$A_k G_k(0; z_0, ..., z_{n-1}, 0, ..., 0)$$

is real and nonnegative. Then

$$T_n(u) = \sum_{k=n}^{\infty} A_k G_k(0; z_0, ..., z_{n-1}, 0, ..., 0).$$

Let

$$B(z) = B(z; n, u) = \sum_{k=n}^{\infty} A_k G_k(z; z_0, ..., z_{n-1}, 0, ..., 0).$$

Then $B(0) = T_n(u)$, and

$$B^{(j)}(z) = \sum_{k=n}^{\infty} A_k G_{k-j}(z; z_j, ..., z_{n-1}, 0, ..., 0), \qquad 0 \leqslant j < n,$$

so that $B^{(j)}(z_j) = 0$, $0 \leqslant j < n$. For $j \geqslant n$ one has

$$B^{(j)}(z) = \sum_{k=j}^{\infty} A_k G_{k-j}(z; 0, \ldots, 0),$$

so that $| B^{(j)}(0)| = | A_j | = u^j$.

COROLLARY 2. $T_n(W) \geqslant 1$, $n = 1, 2, 3, \ldots$.

Proof. In Theorem 4, take $u = W$ and $f = \mathscr{W}$.

4. The Functions T_n

In the proof of Theorem 4 it was tacitly assumed that the series whose maximum defines $T_n(u)$ converges for all values of u, and, for fixed u, converges uniformly in the variables $z_0, \ldots, z_{n-1}$ when they are restricted to $| z | = 1$. This is an easy consequence of Lemma 6.

Restated in terms of W, Lemma 6 asserts that

$$| G_k(0; z_0, \ldots, z_{n-1}, 0, \ldots, 0)| < \frac{e^W - 1}{W^n(k - n + 1)!} \tag{4.1}$$

for $k \geqslant n$ and $| z_j | \leqslant 1$, $0 \leqslant j < n$. Therefore

$$\begin{aligned}
&\sum_{k=n}^{\infty} u^k \, | G_k(0; z_0, \ldots, z_{n-1}, 0, \ldots, 0)| \\
&< u^n \, | G_n(0; z_0, \ldots, z_{n-1})| + \sum_{k=n+1}^{\infty} \frac{u^k}{W^n} \frac{e^W - 1}{(k - n + 1)!} \\
&\leqslant u^n H_n + (e^W - 1) \left(\frac{u}{W}\right)^n \sum_{k=n+1}^{\infty} \frac{u^{k-n}}{(k - n + 1)!} \\
&= u^n H_n + \left(\frac{u}{W}\right)^n \{e^W - 1\} \left\{\frac{e^u - 1}{u} - 1\right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
T_n(u) &< u^n H_n + \left(\frac{u}{W}\right)^n \{e^W - 1\} \left\{\frac{e^u - 1}{u} - 1\right\} \\
&\leqslant \left(\frac{u}{W}\right)^n \{e^W - 1\} \left\{\frac{e^u - 1}{u}\right\},
\end{aligned} \tag{4.2}$$

since $H_n \leqslant W^{-n}$. In the other direction we have

$$T_n(u) = \max \sum_{k=n}^{\infty} u^k \mid G_k(0;\ z_0, \ldots, z_{n-1}, 0, \ldots, 0)|$$

$$\geqslant \max u^n \mid G_n(0; z_0, \ldots, z_{n-1})| = u^n H_n ,$$

so that

$$u^n H_n \leqslant T_n(u) < (u/W)^n \{e^W - 1\}\{e^u - 1\}/u. \tag{4.3}$$

It is easily verified that $T_n(u)/u^n$ is a nondecreasing function of u. If $0 < u < v$,

$$T_n(u) \leqslant (u/v)^n\, T_n(v). \tag{4.4}$$

Therefore $T_n(u)$ is strictly increasing; it follows that there is exactly one positive number u_n which satisfies $T_n(u_n) = 1$.

LEMMA 7. $1 \leqslant T_n(W) < 1.6$.

Proof. The first inequality is Corollary 2; for the second, we have from (4.2) that

$$T_n(W) < W^n H_n + \{e^W - 1\} \left\{\frac{e^W - 1}{W} - 1\right\}$$

$$\leqslant 1 + \{e^W - 1\} \left\{\frac{e^W - 1}{W} - 1\right\}.$$

Since $W < .7378$, we obtain $T_n(W) < 1.6$.

THEOREM 5. $u_n \leqslant W < u_n(1.6)^{1/n}$.

Proof. Lemma 7 and (4.4) with $u = u_n$ and $v = W$.

THEOREM 6. $(.4)^{1/n} H_n^{-1/n} < W \leqslant H_n^{-1/n}$.

Proof. It follows from the proof of Lemma 7 that

$$1 \leqslant T_n(W) < W^n H_n + 0.6.$$

Therefore $1 \geqslant W^n H_n > .4$, and

$$(.4)^{1/n} H_n^{-1/n} < W \leqslant H_n^{-1/n}.$$

Our bounds on $T_n(W)$ together with the functions $B_n(z) = B(z; n, W)$ of

Theorem 4 yield a second method for obtaining a Whittaker function. For these functions, we have the estimates

$$1 \leqslant T_n(W) = B_n(0) < 1.6,$$

$$| B_n^{(j)}(0)| \leqslant W^j T_{n-j}(W) < (1.6)\, W^j,$$

if $1 \leqslant j < n$, and $| B_n^{(j)}(0)| = W^j$ for $j \geqslant n$. It follows that the sequence $\{B_n\}_1^\infty$ is uniformly bounded on bounded sets. Furthermore, every uniform limit of a subsequence of $\{B_n\}$ is a Whittaker function.

5. Theorem B and Related Results

If in Theorem 4 we drop the hypothesis $| f^{(k)}(0)| \leqslant u^k$ for $k \geqslant n$, the same argument yields the following result.

THEOREM 7. *Suppose n is a positive integer and that f is analytic in $| z | \leqslant 1$. If each of $f, f', \ldots, f^{(n-1)}$ has a zero in $| z | \leqslant 1$, then*

$$| f(0)| \leqslant \max \sum_{k=n}^{\infty} | f^{(k)}(0)|\, | G_k(0; z_0, \ldots, z_{n-1}, 0, \ldots, 0)|, \tag{5.1}$$

where the maximum is taken over all sequences $\{z_k\}_0^{n-1}$ whose terms lie on $| z | = 1$. Furthermore, there are functions f for which equality holds in (5.1).

This bound on $| f(0)|$, while best possible, unfortunately is too complicated to be of much use. To obtain something useful from (5.1), we use (4.1) to estimate the second factors on the right. We then have

$$\begin{aligned} | f(0)| &\leqslant \sum_{k=n}^{\infty} | f^{(k)}(0)| \frac{e^W - 1}{W^n (k - n + 1)!} \\ &= \frac{e^W - 1}{W^n} \sum_{m=0}^{\infty} \frac{| f^{(n+m)}(0)|}{(m + 1)!} \\ &\leqslant \frac{1.1}{W^n} \sum_{m=0}^{\infty} \frac{| f^{(n+m)}(0)|}{(m + 1)!}, \end{aligned}$$

which is the inequality of Theorem B. To complete the proof of Theorem B, we take f to be the function $\mathscr{W}$ of Section 3. For this choice of f, one has

$$f(0) = 1$$

and

$$\frac{1}{W^n}\sum_{m=0}^{\infty}\frac{|f^{(n+m)}(0)|}{(m+1)!} \leqslant \sum_{m=0}^{\infty}\frac{W^m}{(m+1)!} = \frac{e^W - 1}{W}.$$

In this case the inequality

$$|f(0)| \leqslant \frac{C}{W^n}\sum_{m=0}^{\infty}\frac{|f^{(n+m)}(0)|}{(m+1)!}$$

will be false if the constant C satisfies

$$C < W/(e^W - 1),$$

and, in particular, if $C = .67$, which completes the proof of Theorem B.

THEOREM 8. *The function* $\mathscr{W}$ *of Theorem* 3 *satisfies*

$$|\mathscr{W}^{(n)}(0)| > (.4)\, W^n, \qquad n = 1, 2, 3, \ldots .$$

Proof. Applying Theorem B to $\mathscr{W}$, one obtains

$$\begin{aligned} 1 &\leqslant \frac{1.1}{W^n}\sum_{m=0}^{\infty}\frac{|\mathscr{W}^{(n+m)}(0)|}{(m+1)!} \\ &\leqslant \frac{1.1}{W^n}|\mathscr{W}^{(n)}(0)| + \frac{1.1}{W^n}\sum_{m=1}^{\infty}\frac{W^{n+m}}{(m+1)!} \\ &= \frac{1.1}{W^n}|\mathscr{W}^{(n)}(0)| + (1.1)\left\{\frac{e^W - 1 - W}{W}\right\} \\ &< \frac{1.1}{W^n}|\mathscr{W}^{(n)}(0)| + .54, \end{aligned}$$

from which the result follows.

6. UNIVALENT DERIVATIVES

For the proof of Theorem C we require the following result.

THEOREM 9. *Suppose n is a positive integer and f is analytic in $|z| \leqslant 1$. If none of $f, f', \ldots, f^{(n-1)}$ is univalent in $|z| \leqslant 1$, then*

$$|f'(0)| \leqslant \frac{1.1}{W^n}\sum_{m=1}^{\infty}\frac{|f^{(n+m)}(0)|}{m!}.$$

Proof. In (3.3), replace f by f' and let $z = 0$. This yields

$$f'(0) = \sum_{k=0}^{n-1} f^{(k+1)}(z_k)\, G_k(0; z_0, ..., z_{k-1})$$

$$+ \sum_{k=n}^{\infty} f^{(k+1)}(0)\, G_k(0; z_0, ..., z_{n-1}, 0, ..., 0), \tag{6.1}$$

which holds for all sequences $\{z_k\}_0^{n-1}$ whose terms lie in $|z| \leqslant 1$. The absolute value of the second sum in (6.1) may be estimated by the method used in the proof of Theorem 7; this absolute value does not exceed

$$M = \frac{1.1}{W^n} \sum_{m=1}^{\infty} \frac{|f^{(n+m)}(0)|}{m!}.$$

If each of the functions $f, f', ..., f^{(n-1)}$ fails to be univalent by having its derivative take the value 0, we can choose the points $\{z_k\}_0^{n-1}$ so that the first sum in (6.1) vanishes. In this case $|f'(0)| \leqslant M$, and we are through. Since this is, in general, not the case, we must eliminate the terms in the first sum by a judicious choice of integrations. We consider two cases.

Case 1. There are points α and β $(\neq \alpha)$ in $|z| \leqslant 1$ such that $f^{(n-1)}(\alpha) = f^{(n-1)}(\beta)$. In this case, integrate both sides of (6.1) from α to β with respect to z_{n-1} and divide by $\beta - \alpha$. We then have

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta f'(0)\, dz_{n-1} = f'(0),$$

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta \sum_{k=0}^{n-1} f^{(k+1)}(z_k)\, G_k(0; z_0, ..., z_{k-1})\, dz_{n-1}$$

$$= \sum_{k=0}^{n-2} f^{(k+1)}(z_k)\, G_k(0; z_0, ..., z_{k-1}),$$

and

$$\left| \frac{1}{\beta - \alpha} \int_\alpha^\beta \sum_{k=n}^{\infty} f^{(k+1)}(0)\, G_k(0; z_0, ..., z_{n-1}, 0, ..., 0)\, dz_{n-1} \right| \leqslant M.$$

The last inequality is obtained by performing the integration along the straight segment from α to β and noting that the modulus of the integrand does not exceed M.

Case 2. There is a point α in $|z| \leqslant 1$ such that $f^{(n)}(\alpha) = 0$. In this case, take $z_{n-1} = \alpha$ in (6.1). This yields

$$f'(0) = \sum_{k=0}^{n-2} f^{(k+1)}(z_k)\, G_k(0; z_0, \dots, z_{k-1})$$

$$+ \sum_{k=n}^{\infty} f^{(k+1)}(0)\, G_k(0; z_0, \dots, z_{n-2}, \alpha, 0, \dots, 0).$$

In either case, we can reduce (6.1) to

$$'(0) = \sum_{k=0}^{n-2} f^{(k+1)}(z_k)\, G_k(0; z_0, \dots, z_{k-1}) + K_1(z_0, z_1, \dots, z_{n-2}), \qquad (6.2)$$

where K_1 is analytic in each variable and satisfies

$$\max |K_1(z_0, z_1, \dots, z_{n-2})| \leqslant M,$$

the maximum being taken over all sequences $\{z_k\}_0^{n-2}$ whose terms lie in $|z| \leqslant 1$.

If we use the same process on (6.2), we obtain

$$f'(0) = \sum_{k=0}^{n-3} f^{(k+1)}(z_k)\, G_k(0; z_0, \dots, z_{k-1}) + K_2(z_0, \dots, z_{n-3})$$

with the same bound on the modulus of K_2. Continuing in the same fashion, we obtain finally

$$f'(0) = K_n,$$

where K_n is a constant which satisfies $|K_n| \leqslant M$. This completes the proof.

Proof of Theorem C. Without loss of generality we can take D to be the disc $|z| \leqslant 1$. Suppose that g is an entire function with $\tau(g) < W$, and that N is a positive integer such that if $j \geqslant N$ then $g^{(j)}$ is not univalent in $|z| \leqslant 1$. If we let $f = g^{(j)}$ (for $j \geqslant N$) we have, from Theorem 9,

$$|f'(0)| \leqslant \frac{1.1}{W^n} \sum_{m=1}^{\infty} \frac{|f^{(n+m)}(0)|}{m!}$$

for all positive integers n.

Choose u so that $\tau(g) = \tau(f) < u < W$. For all large n we have

$$|f^{(n+m)}(0)| < u^{n+m}, \qquad m = 0, 1, 2, \dots .$$

Therefore

$$|f'(0)| < (1.1) \left(\frac{u}{W}\right)^n \{e^u - 1\}$$

for all n sufficiently large. Letting $n \to \infty$, we obtain $f'(0) = g^{(j+1)}(0) = 0$. Since this is true for all $j \geqslant N$, it follows that g is a polynomial of degree at most N, and the proof is complete.

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